Is quantum mechanics a new theory of probability?

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Abstract

Some physicists but many philosophers believe that standard Hilbert space quantum mechanics faces a serious measurement problem, whose solution requires a new theory or at least a novel interpretation of standard quantum mechanics. Itamar Pitowsky did not. Instead, he argued in a major paper [Pitowski 2006] that quantum mechanics offers a new theory of probability. In these and other respects his views paralleled those of QBists (quantum Bayesians): but their views on the objectivity of measurement outcomes diverged radically. Indeed, Itamar's view of quantum probability depended on his subtle treatment of the objectivity of outcomes as events whose collective structure underlay this new theory of probability. I've always been puzzled by the thesis that quantum mechanics requires a new theory of probability, as distinct from new ways of calculating probabilities that play the same role as other probabilities in physics and daily life. In this paper I will try to articulate the sources of my puzzlement. I'd like to be able to think of this paper as a dialog between Itamar and me on the nature and application of quantum probabilities. Sadly, that cannot be: by taking his part in the dialog I will inevitably impose my own distant, clouded perspective on his profound and carefully crafted thoughts.

1 Introduction

Itamar Pitowsky (2003, 2006) developed and defended the thesis that the Hilbert space formalism of quantum mechanics is just a new kind of probability theory. In developing this thesis he argued that all features of quantum probability, including the Born probability rule, can be derived from rational probability assignments to finite "quantum gambles". In defense of his view he argued that all experimental aspects of entanglement, including the Bell inequalities, are explained as natural outcomes of the resulting probability structure. He further maintained that regarding the quantum state as a book-keeping device for quantum probabilities dissolves the BIG measurement problem that afflicts those who believe the quantum state is a real physical state whose evolution is always governed by a linear dynamical law; and that a residual small measurement problem may then be resolved by appropriate attention to the great practical difficulties attending measurement of any observable on a macroscopic system incompatible with readily executable measurements of macroscopic observables.

This is a bold view of quantum mechanics. It promises to transcend not only the frustratingly inconclusive debate among so-called realist interpretations (including Bohmian, Everettian, and "collapse" theories) but also the realist/instrumentalist dichotomy itself. In this way it resembles the pragmatist approach I have been developing myself ([Healey 2012], [Healey 2017]). Our views agree on more substantive issues, including the functional roles of quantum states and probabilities and the consequent dissolution of the measurement problem. But I remain puzzled by Itamar's central thesis. The best way to explain my puzzlement may be to show how the points on which we agree mesh with a contrasting conception of quantum probability. On this rival conception, quantum mechanics requires no new probability theory, but only new ways of calculating probabilities with the same formal features and the same role as other probabilities in physics and daily life.[1]

The rest of this contribution proceeds as follows. In the next section I say what Pitowsky meant by a probability theory and explain how he took quantum probability theory to differ formally from classical probability theory. The key difference arises from the different event structures over which these are defined. Whereas classical probabilities are defined over a σ-algebra of subsets of a set, quantum probabilities are defined over the lattice $\mathcal{L}$ of closed subspaces of a Hilbert space. Gleason's theorem plays a major role here: for a Hilbert space of dimension greater than 2, it excludes a classical truth-evaluation on $\mathcal{L}$ but at the same time completely characterizes the class of possible quantum probability measures on $\mathcal{L}$. Pitowsky sought to motivate the formal difference between quantum and classical probability theory by an extension of a Dutch book argument offered in support of a coherence requirement on rational degrees of belief (in the tradition of Ramsey (1926) and De Finetti (1937)) to a class of what he called "quantum gambles" associated with possible measurements on a quantum system. Section 3 explains the extension but questions how well it justifies the proposed formal modification. A key issue here concerns the independence of the quantum probability of an event from the context in which a non-maximal observable containing it is measured.

A quantum gamble may be settled only if there is a viable procedure for determining which possible event assigned a quantum probability has occurred. So quantum probabilities are not assigned to past events of which we have only a partial record, or no record at all. This restricts "matters of fact" to include only observable records. Section 4 notes how this is in tension with applications of quantum theory in situations where we have no observable records. It goes on to consider hypothetical situations to which quantum theory may be applied in which records may be observable by some but not other observers before these records are permanently erased so they can no longer be read by any observers.

[1] That quantum mechanics involves a new way of calculating probabilities is a point made long ago by Feynman, as Pitowsky himself noted. But Feynman (1951) also says that the concept of probability is not thereby altered in quantum mechanics. Here I am in agreement with Feynman, though the understanding he offers of that concept is disappointingly shallow.

In section 5 I defend an alternative pragmatist view of probability, explaining how it makes room for a notion of objective probability capable of rationally constraining the credences of a rational agent. On this view probabilities in quantum theory are defined over a family of Boolean algebras rather than a single non-Boolean lattice. This is because Born probabilities are defined over possible outcomes of a physical process represented in one, rather than some other, classical event space. Models of decoherence may provide a guide as to what that process is, even though they do not describe the dynamical evolution of a physical quantum state. Any agent who accepts quantum theory should adjust his or her credences to the Born probabilities given by the correct quantum state for one in that physical situation. Insofar as an agent's physical situation constrains what information is available, differently situated agents are correct to assign different quantum states and to adjust their credences accordingly. Each agent should update that quantum state (and associated Born probabilities) on accessing newly available information. This is one way a quantum state provides a book-keeping device for probabilities: the other is provided by its linear evolution while the system remains undisturbed.

2 Quantum Measure Theory

Pitowsky maintained that quantum probability theory differs from classical probability theory formally as well as conceptually. Following Kolmogorov, probability is usually characterized formally as a unit-normed, countably additive, measure Pr on a σ-algebra Σ of subsets E_i $(i \in \mathbb{N})$ of a set Ω: that is

$$\begin{array}{rcll} \Pr & : & \Sigma \longrightarrow [0,1] \text{ satisfies} & \\ 1.\ \Pr(\Omega) & = & 1 & \text{(Probability measure)} \\ 2. \Pr \bigcup\limits_i E_i & = & \sum\limits_i \Pr E_i \text{ provided that } \forall i \neq j\ (E_i \cap E_j = \emptyset) & \end{array}$$

Here Σ forms a lattice which is complemented and distributive, and hence a Boolean algebra. $\Pr(E_i)$ is the probability of the event E_i. Condition 2 is sometimes weakened to finite additivity.

The closed subspaces $\{S_i\}$ of a Hilbert space H also form a lattice $\mathcal{L}(H)$ whose meet $S_i \wedge S_j$ is $S_i \cap S_j$, and whose join $S_i \vee S_j$ is the smallest closed subspace containing all elements of S_i and S_j. H is the maximal element of $\mathcal{L}(H)$ (denoted by $\mathbf{1}$) and the null subspace $\varnothing$ is the minimal element, denoted by $\mathbf{0}$. Each element S has a complement $S^{\perp}$ consisting of every vector orthogonal to every vector in S. Indeed, $S^{\perp}$ is the unique orthocomplement of S. But this lattice is not distributive and so $\mathcal{L}(H)$ is not a Boolean algebra.

A (unit-normed) quantum measure μ on $\mathcal{L}(H)$ is defined as follows:

$$\begin{array}{rcll} \mu & : & \mathcal{L}(H) \longrightarrow [0,1] \text{ satisfies} & \\ 1.\quad \mu(\mathbf{1}) & = & 1 & \text{(Quantum measure)} \\ 2. \mu(\bigvee\limits_i S_i) & = & \sum\limits_i S_i \text{ provided that } \forall i \neq j\ (S_i \perp S_j) & \end{array}$$

Condition 2 may be weakened to finite additivity.

The formal analogy between the conditions defining a probability measure and those defining a quantum measure motivated Pitowsky's proposal to interpret μ as a new kind of *quantum probability*. Accordingly, he referred to the closed subspaces of H as events, or possible events, or possible outcomes of experiments. He also took himself to be following a tradition that takes $\mathcal{L}(H)$ as the structure representing the"elements of reality" in quantum theory. Regardless of that tradition, others have also referred to quantum measures on non-Boolean lattices associated with quantum mechanics as quantum probability functions or measures [Earman 2018], [Ruetsche and Earman 2012]. The set of bounded self-adjoint operators on H forms a von Neumann algebra $\mathcal{B}(H)$: this includes as a subset the set $\mathcal{P}(\mathcal{B}(H))$ of projection operators $\hat{E}$ on H. By virtue of the one-one correspondence between the set of closed subspaces $\{S\}$ of H and the operators $\{\hat{E}_S\}$ that project onto them, $\mathcal{P}(\mathcal{B}(H))$ also forms a lattice isomorphic to $\mathcal{L}(H)$. This generalizes to von Neumann algebras $\mathcal{A}$ other than $\mathcal{P}(\mathcal{B}(H))$ that figure in what Ruetsche (2011) calls extraordinary quantum mechanics or QM_∞.

Gleason's theorem [Gleason 1957] completely characterizes the class of quantum measures on the lattice of closed subspaces of a Hilbert space of dimension greater than 2.

> *Gleason's Theorem:* If H is a (separable) Hilbert space of dimension greater than 2 and μ is a (unit-normed) quantum measure on $\mathcal{L}(H)$, then there is a positive self-adjoint operator $\hat{W}$ of trace 1 such that, for every element $\hat{E}_S$ of $\mathcal{P}(\mathcal{B}(H))$, $\mu(S) = Tr(\hat{W}\hat{E}_S)$.

This theorem has two important consequences. The first is that there are no dispersion-free measures on $\mathcal{L}(H)$ if $\dim(H) \geqslant 3$, where a quantum measure μ is dispersion-free if and only if its range is the set $\{0, 1\}$. This is important since any truth-valuation on the set of all propositions stating that it is event S or instead event $S^\perp$ that represents an element of reality would have to give rise to a dispersion-free quantum measure on $\mathcal{L}(H)$. This, of course, is why Gleason's theorem and similar results (e.g. [Kochen and Specker 1967], [Bell 1966] are considered important "no-hidden variable" results. While it may be understood as the outcome of an experiment, an event S may not be (uniformly) understood as simply the independently existing state of affairs that experiment serves to reveal.

The second important consequence of Gleason's theorem is that every (unit-normed) quantum measure on $\mathcal{L}(H)$ ($\dim(H) \geqslant 3$) uniquely extends to the Born probability measure associated with a quantum state, as represented by a vector in or density operator on H. Indeed, this is true even if the quantum measure is merely finitely additive. This consequence is not surprising, since the Born rule may be stated in the form

$$\Pr{}_{\hat{W}}(A \in \Delta) = Tr(\hat{W}\hat{E}(\Delta)) \qquad \text{(Born rule)}$$

where the quantum state is represented by density operator $\hat{W}$ and $\hat{E}(\Delta)$ is the relevant projection operator from the spectral family defined by the unique self-adjoint operator $\hat{A}$ corresponding to dynamical variable A.

Cognizant of the first consequence of Gleason's theorem, Pitowsky did not defend the radical thesis that quantum theory shows the world obeys a non-classical logic. But he took the second consequence as "one of the strongest pieces of evidence in support of the claim that the Hilbert space formalism is just a new kind of probability theory" ([Pitowski 2006], p.222).

3 Quantum Gambles

Pitowsky developed his conception of quantum probability within the Bayesian tradition pioneered by Ramsey (1926) and De Finetti (1937). This tradition locates probabilities in an agent's rationalized degrees of belief. A necessary, though possibly insufficient, condition for such degrees of belief to be rational is that they be (what has come to be called) coherent.

In this tradition, belief and its degrees are dispositions manifested not by self-avowal but in actions. The condition of coherence is understood in terms of the agent's dispositions to accept a set of wagers at various odds offered by a hypothetical bookie. A set of degrees of belief is coherent just in case it corresponds to a set of such dispositions that is not guaranteed to result in a loss if collectively manifested no matter what the outcome of all the bets in the set. Degrees of belief that are not coherent are said to allow a Dutch book to be made against the agent who has them. So-called synchronic Dutch book theorems are then taken to show that any coherent set of degrees of belief in a set of propositions may be represented by a (classical) probability measure over them.

Following Lewis (1980) an agent's coherent set of degrees of belief are called his or her *credences*. For a subjectivist like De Finetti, there is no further notion of *objective* probability at which a rational agent's credences should aim. Lewis (1980) disagreed. Instead he located a distinct concept of objective probability he called *chance*, of which all we know is that it provides a further rational constraint on an agent's credences through what he called the *Principal Principle*. I defer further consideration of this principle until section 5, since it plays no role in Pitowsky's own view of quantum probability theory.

According to Pitowsky (2006), a quantum gamble consists of four steps:

> 1. A *single* physical system is prepared by a method known to everybody.
>
> 2. A finite set $\mathcal{M}$ of incompatible measurements, each with a finite number of possible outcomes, is announced by the bookie. The agent is asked to place bets on the possible outcomes of each one of them.
>
> 3. One of the measurements in the set $\mathcal{M}$ is chosen by the bookie and the money placed on all other measurements is promptly returned to the agent.

4. The chosen measurement is performed and the agent gains or loses in accordance with his bet on that measurement.

Here $\mathcal{M}$ is identified with a set of Boolean algebras $\{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_k\}$, each generated by the possible outcomes in $\mathcal{L}$ of the measurement to which it corresponds. The elements of $\mathcal{B} = \{S_1, S_2, ...S_m\} \in \mathcal{M}$ will not all be one-dimensional subspaces if $\mathcal{M}$ is not a maximal measurement. Pitowsky (2006, p.223) maintained that "by acting according to the standards of rationality the gambler will assign probabilities to the outcomes". He took the gambler in question to recognize identities in the logical structure consisting of the outcomes in $\mathcal{L}$, and in particular the cases in which the same outcome is shared by more than one experiment (i.e. type of measurement in $\mathcal{M}$.) But, crucially, this gambler was *not* assumed to know quantum mechanics.

Pitowsky then argued (2006, p.227: Corollary 7) that any such quantum gambler *not* meeting these standards of rationality must assign probability values to elements of a finite sublattice $\Gamma_0 \subset \mathcal{L}(H)$ $(\dim(H) \geq 3)$ that cannot be extended to a quantum measure on a finite $\Gamma \supset \Gamma_0$. He took this conclusion to establish the claim that quantum theory is a new theory of probability

Notice that this argument is not offered as a derivation of the Born rule insofar as it does not mention the quantum state $\hat{W}$ used in stating that rule. It concludes only that a rational agent's credences should be consistent with the probabilities specified by quantum mechanics through application of the Born rule to *some* quantum state. Instead, this conclusion is presented as justification for the claim that quantum probability is quantum measure theory: but is it warranted?

How do the standards of rationality constrain a quantum gambler's assignments of "probabilities" (i.e. degrees of belief) to outcomes in $\mathcal{L}$? Assume that for every $\mathcal{B} \in \mathcal{M}$ such an agent assigns degree of belief $cr(S|\mathcal{B})$ to outcome S of a measurement corresponding to $\mathcal{B}$. Pitowsky took the standards of rationality to require the agent to assign degrees of belief in accordance with two rules (in my notation):

RULE 1: For each measurement $\mathcal{B} \in \mathcal{M}$ the function $cr(\bullet|\mathcal{B})$ is a probability distribution on $\mathcal{B}$.

RULE 2: If $\mathcal{B}_1$, $\mathcal{B}_2 \in \mathcal{M}$, and $S \in \mathcal{B}_1 \cap \mathcal{B}_2$ then $cr(S|\mathcal{B}_1) = cr(S|\mathcal{B}_2)$.

Rule 1 is imposed to insure the coherence of the agent's degrees of belief concerning the possible outcomes of a single measurement: If the agent's degrees of belief do not confirm to this rule they will dispose him or her to accept bets on these outcomes guaranteed to lead to a sure loss. This is the standard Dutch book argument as to why a rational agent's degrees of belief must be representable as a (finitely additive) probability measure.

The force of such Dutch book arguments has been a topic of extended debate among philosophers of probability, and alternative arguments have been offered as to why a rational agent's degrees of belief should be representable as probabilities. Briefly stated, here are three standard objections seeking to undermine the Dutch book argument: A rational agent may simply refuse to bet: his betting behavior may fail to reveal his degrees of belief insofar as it is a function of the rest of his cognitive and affective state: agents have degrees of belief in propositions whose truth-value cannot be determined because there is no corresponding settleable outcome. I will not press either of the first two objections now. Pitowsky did anticipate this third objection by requiring that there be a viable procedure for determining which possible event assigned a quantum probability has occurred, and I will pursue this issue in section 4.

Rule 2 requires a quantum gambler's credences to be non-contextual, in the sense that he or she have the same degree of belief in the truth of propositions stating the outcome of a measurement of a non-maximal observable no matter what type of measurement led to that outcome. It is a remarkable fact about the Born rule of quantum mechanics that the probabilities it yields are non-contextual in this sense. But Pitowsky's gambler cannot be assumed to know quantum mechanics. So why should the quantum gambler's credences be non-contextual?

Pitowsky (2006, p.216) first addresses this question in section 2.1 as follows (emphases in the original):

> ...the *identity* of events which is encoded by the structure also involves judgments of probability in the sense that *identical events always have the same probability*. This is the meaning of accepting a structure as an algebra of events *in a probability space*.

Here he takes the structure of events to be the lattice $\mathcal{L}(H)$ of subspaces of some Hilbert space H whose Boolean sublattices correspond to measurements on a quantum system, some incompatible with others. Application of quantum theory's Born rule turns $\mathcal{L}(H)$ into a quantum measure space when quantum state $\hat{W}$ generates a subspace measure μ through $\mu(S) = Tr(\hat{W}\hat{P}_S)$, where $\hat{P}_S$ is the (unique) projection operator onto subspace $S \in \mathcal{L}(H)$. But so far we have been offered no reason to regard $\mathcal{L}(H)$ as a *probability* space.

The Dutch book argument Pitowsky took to require a rational quantum gambler's credences to conform to Rule 1 does not require that agent's credences to conform to Rule 2. To the extent that argument is successful, it establishes only that each Boolean subalgebra $\mathcal{B}$ of $\mathcal{L}(H)$ corresponding to a measurement in $\mathcal{M}$ may be taken to define a σ-algebra of subsets of $\mathbb{R}$ representing possible outcomes of that measurement, and in that sense $\mathcal{B}$ is a (subjective) probability space. Additional argument is needed to justify the claim that the full lattice $\mathcal{L}(H)$ is a probability space.

The argument Pitowsky (2006, p.216) offers in section 2.1 proceeds by analogy to a classical application of probability theory to games of chance:

> Consider two measurements A, B, which can be performed together; and suppose that A has the possible outcomes $a_1, a_2, ..., a_k$, and B the possible outcomes $b_1, b_2, ..., b_r$. Denote by $\{A = a_i\}$ the event "the outcome of the measurement of A is a_i", and similarly for $\{B = b_j\}$. Now consider the identity:

$$\{B = b_j\} = \bigcup_i^k (\{B = b_j\} \cap \{A = a_i\}) \tag{1}$$

> This is the distributivity rule which holds in this case as it also holds in all classical cases. This means, for instance, that if A represents the roll of a die with six possible outcomes and B the flip of a coin with two possible outcomes, then Eq.(1) is trivial. Consequently the probability of the left hand side of Eq.(1) equals the probability of the right hand side, for every probability measure.

In a classical joint probability space the event $\{B = b_j\}$ occurs if and only if some outcome $\{B = b_j\} \cap \{A = a_i\}$ occurs, and so the identity Eq.(1) holds trivially. If instead the event $\{B = b_j\}$ is the outcome of a trial with a set of possible outcomes $\{\{B = b_n\} : n = 1, ..., N\}$, then it is not trivial that the probability of the left hand side of Eq.(1) equals the probability of the right hand side, since these concern probabilities in different trials with different probability spaces. We would be astonished if a fair coin always came up heads whenever a die was rolled at the same time, but that is because we have empirical reasons to discount the influence of dice rolls on the outcomes of simultaneous coin tosses.

Pitowski draws an analogy between Eq.(1) (as applied to compatible quantum measurements or coin flips and dice rolls) and Eq.(2):

$$\bigcup_i^k (\{B = b_j\} \cap \{A = a_i\}) = \{B = b_j\} = \bigcup_i^l (\{B = b_j\} \cap \{C = c_i\}) \tag{2}$$

In this equation, A, B, C, are quantum observables such that $[A, B] = 0$, and $[B, C] = 0$ but $[A, C] \neq 0$, and $c_1, c_2, ..., c_l$ are the possible outcomes of C. Since A, C are incompatible observables they are not jointly measurable. Accordingly, the Born rule of quantum mechanics does not assign joint probabilities to events such as $\{A = a_m\}$, $\{C = c_n\}$. Unlike the events $\{A = a_m\}, \{B = b_n\}$ in Eq.(1), such events are not elements of any (classical) probability space acknowledged by quantum mechanics. Indeed, as Fine (1982) showed, the joint Born probabilities for a some sets of pairwise compatible observables are not marginals of *any* (classical) joint probability distribution. Eq.(2) expresses a trivial identity in a lattice $\mathcal{L}(H)$ if $\cup, \cap$ are read as join and meet operations in the lattice. But they cannot generally be read as set-theoretic union and intersection of elements of a σ-algebra of subsets of a set of outcomes of a (classical) probability space.

In a way, this was Pitowsky's point. He used the analogy only to motivate his proposal that quantum probability is *different* from classical probability in just this way. As he put it ([Pitowski 2006], pp.216-7)

> I assume that the **0** of the algebra of subspaces represents impossibility (zero probability in all circumstances) **1** represents certainty (probability one in all circumstances), and the identities such as

> Eq.(1) and Eq.(2) represent identity of probability in all circumstances. This is the sense in which the lattice of closed subspaces of the Hilbert space is taken as an algebra of events. I take these judgments to be natural extensions of the classical case; a posteriori, they are all justified empirically.

However, his *argument* for this proposal was supposedly based on rationality requirements on the credences of a quantum gambler ignorant of quantum mechanics, and here empirical considerations are out of place. If probability *just is* rational credence, then a requirement of rationality cannot be based on empirical considerations of the kind that are taken to warrant acceptance of quantum mechanics and the non-contextuality of probabilities consequent on application of the Born rule. Immediately after stating Rule 2, Pitowsky (2006) says that this follows from an identity between events essentially equivalent to Eq. (2), and the principle that identical events in a probability space have equal probabilities. But, as we have seen, that principle applies automatically only when these events are part of a single probability space, corresponding to a single trial or measurement context. A lattice $\mathcal{L}(H)$ is naturally understood as a quantum measure space, but only its Boolean subspaces are naturally understood as *probability* spaces. Pitowsky's appeal to rationality requirements on a quantum gambler does not justify his thesis that quantum theory is a new kind of probability theory.

4 Objective knowledge of quantum events

In the language of probability theory, the term 'event' may be used to refer either to a mathematical object (such as a set) or to a physical occurrence. When arguing that the Hilbert space formalism of quantum mechanics is a new theory of probability, Pitowsky took that theory to consist of an algebra of events, and the probability measure defined on it. Events in this sense are *mathematical* objects—subspaces of a Hilbert space. But to each such object he also associated a class of actual or possible *physical* occurrences—outcomes of an experiment in which a quantum observable is measured. A token physical event e_S occurs just in case in a measurement M of observable O the outcome is o_i, an eigenvalue of $\hat{O}$ with eigenspace S. Here M is a token physical procedure corresponding to the mathematical object $\mathcal{B}_M$, a Boolean subalgebra of a lattice $\mathcal{L}(H)$ of subspaces including S, where $\hat{O}$ is a self-adjoint operator on H.

When a quantum gamble is defined over a set $\mathcal{M}$ of incompatible measurements, each of these is characterized by the corresponding Boolean subalgebra of $\mathcal{L}(H)$. A token event e_S settles a gamble if it is the outcome of an actual measurement procedure M corresponding to the mathematical object $\mathcal{B}_M \in \mathcal{M}$, where the bookie chose to perform M rather than some other measurement in the gamble. Following the subjectivist tradition pioneered by Ramsey, Pitowsky stressed that, for a quantum gamble to reveal an agent's degrees of belief, the outcome of whatever measurement is chosen by the bookie must be settleable.

> A proposition that describes a possible event in a probability space is of a rather special kind. It is constrained by the requirement that there should be a viable procedure to determine whether the event occurs, so that a gamble that involves it can be unambiguously decided. This means that we exclude many propositions. For example, propositions that describe past events of which we have only a partial record, or no record at all. ([Pitowsky 2003], p.217)
>
> Recall that in section 2.2 we restricted "matters of fact" to include only observable records. Our notion of "fact" is analytically related to that of "event" in the sense that a bet can be placed on x_1 only if its occurrence, or failure to occur, can be unambiguously recorded. (*op. cit.*, p.231)

In the previous section I questioned the force of Pitowsky's argument that a rational agent's degrees of belief in the outcomes of quantum gambles will be representable as a quantum probability measure on $\mathcal{L}(H)$. But even if that argument were sound it would not extend to all uses of probability in applications of quantum theory. The Born rule may be legitimately applied to propositions concerning events whose outcomes are unknown or even unknowable. For such events there is no viable procedure to determine what the outcome is, so there is no settleable quantum gamble involving these events. An agent acting according to the standards of rationality associated with a quantum gamble need not assign *any* probabilities to the possible outcomes of such an event: rules 1 and 2 need not apply. Such a rational agent may have degrees of belief concerning these outcomes that cannot be represented as a quantum measure on a corresponding $\mathcal{L}(H)$ even though each outcome has a well-defined Born probability.

Quantum theory is often applied to occurrences in distant spacetime regions from which no observable records are accessible. These include processes in the center of stars (including the cores of neutron stars) and quantum field fluctuations in the early universe. There is no restriction on the application of the Born rule to calculate probabilities of outcomes associated with such processes. Of course these cannot be understood as the outcomes of experiments since no experimenters could have been present in those regions. But the Born rule is commonly applied to yield probabilities of outcomes of processes in which no experiment is involved, whether or not these are referred to as measurements. The notion of measurement is notoriously obscure in quantum mechanics, as Bell forcefully pointed out in his article "Against 'Measurement'". But even there he posed the rhetorical question

> If the theory is to apply to anything but highly idealized laboratory operations, are we not obliged to admit that more or less 'measurement-like' processes are going on more or less all the time, more or less everywhere? ([Bell 2004], p.216)

These days references to measurement are often replaced by talk of decoherence, though it is now generally acknowledged that if there is a serious measurement problem then appeals to decoherence will not solve it. While agreeing with Pitowsky that there is no BIG measurement problem, in the next section I will indicate how quantum models of decoherence may be used as guides to the legitimate application of the Born rule in a pragmatist view of quantum mechanics. But in this view a proposition to which the Born rule assigns a probability does not describe the outcome of a measurement but the physical event of a magnitude taking on a value, whether or not this is precipitated by measurement.

Even when a magnitude (a.k.a. observable) takes on a value recording the outcome of a quantum measurement, that record may be subsequently lost, at which point Pitowsky's view excludes any event described by a proposition about that outcome from a quantum probability space. While such scruples may not be out of place in the context of a quantum gamble, only an indefensible form of verificationism would prohibit retrospective application of the Born rule to that event. So quantum mechanics permits application of the concept of probability in circumstances in which it cannot be understood to be defined on a space of events consistent with Pitowsky's view.

One might object that no record of a measurement outcome is ever irretrievably lost. But recent arguments ([Healey 2018], [Leegwater 2018]) challenge such epistemic optimism in the context of *Gedankenexperimenten* based on developments of the famous Wigner's friend scenario. These arguments threaten the objectivity of measurement outcomes under the assumption that unitary, no–collapse single-world quantum mechanics is applicable at all scales, even when applied to one observer's measurement on another observer's lab (including any devices in that lab recording the outcomes of prior quantum measurements). It is characteristic of these arguments that an observation by one observer on the lab of another completely erases all of the latter's records of the outcome of his or her prior quantum measurement. As Leegwater (2018, p.13) put it

> such measurements in effect erase the previous measurement's result; they must get rid of all traces from which one can infer the outcome of the first measurement.

In this situation, the first observer's measurement outcome was a real physical occurrence, observable by and (then) known to that observer. But the second observer's measurement on the first observer's lab erased all records of that outcome, and neither of these observers, nor any other observer, can subsequently verify what it was. Indeed, even the first observer's memory of his own result has been erased. This is clearly a situation in which Pitowsky would have excluded the event corresponding to the first observer's measurement outcome from any probability space. For in no sense does a true proposition describing that outcome state (what he called) a "matter of fact".

I wonder what Pitowsky would have made of these recent arguments. Leegwater (2018) presents his argument as a "no-go" result for unitary, no–collapse single-outcome (relativistic) quantum mechanics, while Healey (2018) takes the third argument he considers to challenge the objectivity of measurement outcomes. Brukner (2018) formulates a similar argument in a paper entitled "A no-go theorem for observer-independent facts", in which he says

> We conclude that Wigner, even as he has clear evidence for the occurrence of a definite outcome in the friend's laboratory, cannot assume any specific value for the outcome to coexist together with the directly observed value of his outcome, given that all other assumptions are respected. Moreover, there is no theoretical framework where one can assign jointly the truth values to observational propositions of different observers (they cannot build a single Boolean algebra) under these assumptions. A possible consequence of the result is that there cannot be facts of the world per se, but only relative to an observer, in agreement with Rovelli's relative-state interpretation, quantum Bayesianism, as well as the (neo)-Copenhagen interpretation.

Pitowsky founded his Bayesian approach to quantum probability on the notion of a quantum gamble, from which he excluded propositions about measurement outcomes that describe past events of which *we* have only a partial record, or no record at all. This strongly suggests that he would be unwilling to countenance applications of quantum theory to propositions about measurement outcomes stating observer-dependent facts. His tolerance of agent-dependent probabilities did not extend to acquiescence in non-objective facts about their subject matter. The paper by Hemmo and Pitowsky (2007) criticized the way many-worlds interpretations of quantum mechanics understood the use of probability in quantum mechanics. So Pitowsky would almost certainly have rejected the option of evading the conclusion of recent non-go arguments by countenancing multiple outcomes of a single quantum measurement. I speculate that his response to these arguments would have been to follow von Neumann by rejecting the assumption that unitary quantum mechanics may be legitimately applied to the measurement process itself.

5 A pragmatist view of quantum probability

As we saw in section 3, Pitowsky's Bayesian view of quantum probability relied on his Rule 1, which requires an agent's degrees of belief in the possible outcomes of a quantum measurement characterized by a Boolean algebra $\mathcal{B}$ to be a (finitely additive) probability measure on the associated algebra of sets. Here he followed in the tradition of Ramsey and de Finetti, who first employed synchronic Dutch book arguments in support of the probability laws as standards of synchronic coherence for degrees of belief. But each of them also took an agent's betting behavior as a *measure* of his or her degrees of belief, giving operational significance to the numerical probabilities by which these could be represented. Pitowsky quoted Ramsey as follows:

> "The old-established way of measuring a person's belief " by proposing a bet, and seeing what are the lowest odds which he will accept,

is “fundamentally sound”. ([Pitowski 2006], p.223)

If degrees of belief are to be behaviorally defined in terms of betting behavior, then it is essential that these bets be settleable—as Pitowsky insisted. But operational definitions of individual cognitive states like beliefs and desires or preferences are now commonly regarded as inadequate. These are better understood within a broadly functionalist approach to the mind in which the behavioral manifestations of an individual belief are a function of many, if not all, an agent's other cognitive states. Even if there are such things as degrees of belief, these cannot be reliably measured by an agent's betting behavior of the kind that figures in an argument intended to show that a (prudentially) rational agent's degrees of belief will be representable as probabilities.

A Dutch book argument may still be used to justify Bayesian coherence of a set of degrees of belief as a normative condition on epistemic rationality, analogous to the condition that a rational agent's full beliefs be logically compatible. The view that ideally rational degrees of belief must be representable as probabilities has been called *probabilism* ([Christensen 2004], p.107): such degrees of belief are known as credences. The force of a Dutch book argument for probabilism is independent of whether there are any bookies or whether bets in a book are settleable. Understood this way, Rule 1 is justified as a condition of epistemic rationality on degrees of belief in a quantum event, whether or not a gamble that involves it can be unambiguously decided.

Probabilism places only minimal conditions on the degrees of belief of an individual cognitive agent, just as logical compatibility places only minimal conditions on his or her full beliefs. Pitowsky sought to justify additional conditions, sufficient to establish the result that a rational agent's credences in quantum events involving a system be representable by a quantum measure on the lattice $\mathcal{L}(H)$ of subspaces of a Hilbert space H associated with that system (provided that $dim(H) \geqslant 3$). This was the key result he took to justify his claim that the Hilbert space formalism of quantum mechanics is a new theory of probability.

Achieving this result depended on Rule 2: the further condition that a rational agent's credence in a quantum event represented by subspace $S \in \mathcal{L}(H)$ be the same, no matter whether it occurs as an outcome of measurement M_1 (represented by Boolean sub-lattice $\mathcal{B}_1 \subset \mathcal{L}(H)$) or M_2 (represented by $\mathcal{B}_2$). Pitowsky took Rule 2 to follow from the identity between events, and the principle that identical events in a probability space have equal probabilities. As embodied in Eq. (2), Pitowsky took these judgments to be natural extensions of the classical case (as embodied in (1)): and he said of these judgments “a posteriori, they are all justified empirically”.

This justification for Rule 2 is quite different from the justification offered for Rule 1, which was justified not empirically but as a normative requirement on epistemic rationality. As such, Rule 1 showed why an epistemically rational agent should adopt degrees of belief representable as probabilities over a classical probability space of events associated with each Boolean subalgebra $\mathcal{B}$ of $\mathcal{L}(H)$: in that sense, it justified considering $\mathcal{B}$ as a probability space. It is because no analogous normative requirement justifies taking $\mathcal{L}(H)$ itself to be a probability space that Pitowsky appealed instead to empirical considerations. But while such empirical considerations may justify the introduction of a quantum measure μ over $\mathcal{L}(H)$ as a convenient device for generating a posteriori justified objective probability measures over Boolean subalgebras of $\mathcal{L}(H)$, this does not show that μ is itself a probability, or $\mathcal{L}(H)$ a probability space. Only by appeal to some additional normative principle of epistemic rationality could a Bayesian in the tradition of Ramsey and de Finetti attempt to show that.

The bearing of empirical considerations on credences has been a controversial issue among Bayesians. De Finetti maintained that there is nothing more to probability than each agent's actual credences. Ramsey allowed that probability in physics may require more, and in this he has been followed by the physicist Jaynes (2003) and other objective Bayesians. Contemporary QBists portray the Born rule as an empirically motivated additional normative constraint on credences [Fuchs and Schack 2013], while most physicists still follow Feynman in seeking an objective correlate of probability in stable frequencies of outcomes in repeated experimental trials.

Lewis believed that objective probability required the kind of indeterminism generally thought to be manifested by radioactive decay. He took orthodox quantum mechanics to involve such indeterminism, with objective probabilities supplied by the Born rule serving as paradigm instances of what he called *chance*. Lewis (1980) formulated the Principal Principle he took to state all we know about chance by linking this to credence. He later proposed a modification to square it with his Humean metaphysics. But Ismael (2008) argued that the modification was a mistake, and essentially restated his original principle. In Lewis's (1994, pp.227-8) words,

> If a rational believer knew that a chance of [an event e] was 50%, then almost no matter what he might or might not know as well, he would believe to degree 50% that [e] was going to occur. Almost no matter, because if he had reliable news from the future about whether e would occur, then of course that news would legitimately affect his credence.

There is now a large literature on how a Principal Principle should be stated, and how, if at all, it may be justified. This includes Ismael's helpful formulation as an implicit definition of a notion of chance:

> The [modified Lewisian] chance of A at p, conditional on any information H_p about the contents of p's past light cone satisfies: $Cr_p(A/H_p) =_{df} Ch_p(A)$.

Lewis was right to add to credence a second, more objective, concept of probability: but he was wrong to restrict its application to indeterministic contexts. The physical situation of a localized agent frequently imposes limitations on access to relevant information about an event about whose occurrence he or she wishes to form reasonable credences. This occurs, for example, in so-called games of chance and in classical statistical mechanics as well as in the quantum domain. In such situations general principles or a physical theory may provide a reliable way of "packaging" the accessible information in a way that permits generally reliable inferences and appropriate actions. That is how I understand the role of the Born rule in quantum mechanics. To accept quantum mechanics is to grant the Born rule objective authority over one's credences in quantum events and thereby regard it as epistemic expert in this context.

The Born rule associates a set of general probabilities to events of certain types involving a kind of physical system assigned a quantum state. The physical situation of an actual or merely hypothetical agent gives that agent access to information about the surrounding circumstances that may be sufficient to assign a specific quantum state to one or more individual systems. Differently situated agents should sometimes correctly assign different quantum states because different information is accessible to each. This may occur just because the agents do not share a single spacetime location, in conformity to Ismael's modified Lewisian chance: Born probabilities supply many examples of such objective probabilities. After a specific quantum state is assigned to an individual system, the Born rule yields a probability measure over events of certain types involving it. By instantiating the general Born rule, the agent may then derive an objective chance distribution over possible events.

This is a classical probability distribution over an event space with the structure of a Boolean algebra $\mathcal{B}$, not a quantum measure over a non-Boolean lattice: in a legitimate application of the Born rule no probability is assigned to events that are not elements of $\mathcal{B}$. Events assigned a probability in this way are given canonical descriptions of the form $Q_s \in \Delta$, where Q is a dynamical variable (observable), s is a quantum system, and Δ is a Borel set of real numbers: I call $Q_s \in \Delta$ a *magnitude claim*. Some of these events are appropriately redescribed as measurement outcomes: for others, this redescription is less appropriate.

Probability theory had its origins in a dispute concerning dice throws, so it may be helpful to draw an analogy with applications of probability theory to throws of a die. The faces of a normal die are marked in such a way that the point total of opposite faces sums to 7, and the faces marked $4, 6$ meet on one edge of the die. Consider an evenly weighted die that includes a small amount of magnetic material carefully distributed throughout its bulk. This die may be thrown onto a flat, level surface in one or other of two different kinds of experiments: a magnetic field may be switched on or off underneath the surface as the die is thrown. In an experiment with the magnetic field off, a throw of the die is fair, so there is an equal chance of 1/6 that each face will land uppermost. (Notice that, though natural, this use of the term 'chance' does not conform to Lewis's Principal Principle—even in Ismael's modified formulation—since it does not presuppose that dice throws are indeterministic processes.) In an experiment with the magnetic field on, the die is biassed so that some faces are more likely to land uppermost than others. But because of the careful placement of the magnetic material within the die, the chance is still 1/3 that a face marked 4 or 6 lands uppermost.

In each kind of experiment, a probabilistic model of dice-throwing may be tested against frequency data from repeated dice throws of that kind. This model includes general probability statements of the form $P_X(E) = p$ specifying the probability of an event of type E describing a possible outcome of a throw of the die in an experiment of type X. Consider the situation of an actual or hypothetical agent immediately prior to an individual throw of the die. The information accessible to this agent does not include the outcome of that throw: nor could it include all potentially relevant microscopic details of the initial and boundary conditions present in that throw, even if it were a deterministic process. But it does include knowledge of whether the magnetic field is on or off: failure to obtain this relevant, accessible information would be an act of epistemic irresponsibility.

Having accepted a probabilistic model on the evidence provided by relative frequencies of different outcome types in repeated throws of both kinds, this actual or hypothetical agent has reason to instantiate the general probability statement of the form $P_X(E) = p$ for each possible event e of type E in experiment x of kind X. The result is the *chance* of e in x—that to which an epistemically rational agent should match his or her credence concerning event e in experiment x. What that chance is may depend on the experiment x. If e is an event of face 1 landing uppermost, then the chance of e may be less if the magnetic field is on in x than it would have been if the field had been off. But if e' is the event of a prime-numbered face landing uppermost, then the chance of e' is $2/3$ both in experiment x (magnetic field off) and in experiment x' (magnetic field on).

Here is the analogy between this dice-throwing example and quantum probabilities. In both cases there are general probability rules that may be instantiated to give chances to which an (actual or hypothetical) rational agent who accepts those rules should match his or her credences in circumstances of the kind specified by the rule. In both cases these circumstances may be described as those of an experiment, but in neither case is this description essential—no experimenter need be present if the circumstances occur naturally.

In both cases a possible event of a certain type may be assigned the same chance in different circumstances, describable as experiments of different kinds. Both cases feature non-contextual probability assignments to events of the same type. There is no temptation to combine the classical probability spaces corresponding to different kinds of experiment in the dice-throwing example into a single non-classical event space. The rich formal structures to which Pitowsky appealed in the quantum case have no analog in the dice-throwing example. But I remain unpersuaded that the absence of any corresponding structures in the dice-throwing example undermines the force of the analogy.

It is the physical circumstances in which a system finds itself that determine which applications of the Born rule are legitimate (if any), and which are not. Quantum mechanics itself does not specify these circumstances, but physicists have developed reliable practical knowledge of when they obtain. Quantum models of decoherence can provide useful guidance in judging whether and which application of the Born rule is legitimate, by selecting an appropriate Boolean algebra $\mathcal{B}$ of events corresponding to a so-called "pointer basis" of subspaces of the Hilbert space H_s of s. Here s will typically differ from the target system t

to which a quantum state has been assigned in order to apply the Born rule.

Models of decoherence are neither sufficient nor necessary to solve any BIG measurement problem: there is no such problem. That problem would arise only on the mistaken assumption that a quantum state specifies all the physical properties of the system to which it is assigned. But I agree with Pitowsky that a quantum state does not do this: instead, it acts as a book-keeping device for a rational agent's credences by requiring them to conform to the associated Born rule probabilities. Quantum mechanics cannot explain why measurements have definite outcomes, since application of its Born rule presupposes that they do. But since a quantum state does not specify its physical condition, there is no tension between a system's being assigned a superposed state and the truth of a magnitude claim that it has a particular eigenvalue of the corresponding operator.

The Born rule is legitimately applied only to significant magnitude claims. The significance of a magnitude claim is not certifiable in terms of truth-conditions in some kind of "quantum semantics". In my pragmatist view, the significance of any claim arises from its place in an inferential web of statements, ultimately linked to perception and action through practical rather than theoretical inference. This inferentialist "metasemantics" supports an analog rather than digital view of the content of each magnitude claim as coming in degrees rather than making the claim simply meaningful or meaningless. Though extraordinarily rapid, complete, and practically irreversible, decoherence is also a matter of degree in applicable quantum models. It follows that there can be no sharp line dividing meaningful from meaningless magnitude claims, or legitimate from illegitimate applications of the Born rule.

This sheds light on the import of the arguments discussed in the previous section purporting to show that the universal applicability of unitary quantum mechanics is incompatible with the assumption that quantum measurements always have unique, objective outcomes. In my pragmatist view, the outcome of a quantum measurement can be stated in a magnitude claim $Q_s \in \Delta$ about a system s that may be thought of as a measuring apparatus. The claim $\mathbf{Q}_s \in \mathbf{\Delta}$ is true at a time if and only if (the value of) Q_s is then an element of Δ. But $\mathbf{Q}_s \in \mathbf{\Delta}$ derives its content not through this (trivial) truth-condition, but through the reliable inferences into which the claim then enters.

In ordinary circumstances we assume that observation of an experimenter's laboratory records is a reliable way for anyone to determine the outcome of a quantum measurement in that laboratory. Indeed, that is a common understanding of what it is for the measurement to have a unique, objective outcome. But this assumption breaks down in the extreme circumstances of *Gedankenexperimenten* like those that figure in arguments considered in [Healey 2018], [Leegwater 2018]. In such circumstances an observer in an initially physically isolated laboratory will correctly report the outcome of his experiment even though another observer may report a different outcome after subsequently entering that laboratory and making her own observations of its records. Moreover these observers will not then register any disagreement, since all of the first observer's initial records (including memories) will have been erased by the time of the second observation.

In such cases, processes modeled by decoherence confined to the first observer's laboratory rendered reliable a host of inferences based on magnitude claims stating records of his quantum measurement. This endowed these claims with a high degree of significance, so his observations warranted him in taking them truly to state the unique, physical outcome of his measurement. But the extraordinary circumstances of the *Gedankenexperiment* in fact restrict the domain of reliability to the context internal to the laboratory within which these processes were confined. There is a wider context that includes subsequent physical processes coupling that laboratory to a second observer and her laboratory. In that wider context, inferences based on the magnitude claims stating records of his quantum measurement cease to be reliable, thereby curtailing the significance of these claims, which were nevertheless true in the context in which he made them.

In my pragmatist view, quantum measurements would have unique, physical outcomes even in extraordinary circumstances like those described in the *Gedankenexperimenten* that figure in the arguments considered in [Healey 2018], [Leegwater 2018]. The outcomes could be described by true magnitude claims about individual experimenters' physical records of them. Some such claims made by different experimenters may seem inconsistent. But the inconsistency is merely apparent. A correct understanding of these claims requires that their content be relativized to the physical context to which they pertain. So contextualized, each experimenter in one of these *Gedankenexperimenten* can allow the objective truth of the others' reports of each of their unique measurement outcomes while consistently stating the outcome of his or her own measurement.

This implicit limitation on the content of observation reports of the outcomes of quantum measurements may usually be neglected. Quantum decoherence is so pervasive that we will never be able to realize the extraordinary circumstances required by the *Gedankenexperimenten* that figure in the arguments considered in [Healey 2018], [Leegwater 2018]: even a powerful quantum computer would not constitute an agent capable of performing and reporting the outcome of a quantum measurement in a physically isolated laboratory. Because we all inevitably share a single decoherence context, true reports of the unique, physical outcomes of quantum measurements provide the objective data that warrant acceptance of the theory we use successfully to predict their objective probabilities.

6 Conclusion

Quantum mechanics is not a new theory of probability. On the contrary, it constitutes perhaps our most successful deployment of classical probability theory in physics. It is not only the mathematics of probability that are classical here: the concept itself functions in basically the same way in quantum mechanics that it always has. This function is as a source of expert, "pre-packaged" advice to an actual or merely hypothetical situated agent on how strongly to believe statements whose truth-value that agent is not in a position to determine from the accessible information.

Quantum probabilities may be represented by means of quantum measures on non-Boolean lattices, in which case Gleason's theorem offers an elegant characterization of the range of quantum probabilities yielded by applications of the Born rule. But a quantum measure is not a probability measure, and the lattice of closed subspaces of a Hilbert space is not a probability space, despite the non-contextuality of quantum probabilities.

Dutch book arguments for synchronic coherence support an epistemic norm that an agent's degrees of belief in a set of propositions should be representable by a probability measure. But there is a more objective notion of probability than that of an individual agent's actual credences. An agent's credences become subject to additional norms through acceptance of general probabilities: these importantly include the Born probabilities prescribed by a legitimate application of the Born rule. Coherence is an epistemic norm even for beliefs concerning unsettleable events, including the outcomes of quantum 'measurements' that no-one knows, and those not everyone *can* know—but even these outcomes are as objective as science needs them to be.

In all these ways I have come to disagree with Itamar's view of probability in quantum mechanics. I only wish he could now reply to show me why I am wrong: we could all learn a lot from the ensuing debate. Let me finish by reiterating two important points of agreement. There is no BIG quantum measurement problem: there is only the small problem of physically modeling actual measurements within quantum theory and showing why some are much easier than others. A quantum state is not an element of physical reality ($|\psi\rangle$ is not a beable); it is a book-keeping device for updating an agent's credences as time passes or in the light of new information (even if there is no actual agent or bookie to keep the books)!

References

[Bell 1966] Bell, J. S.: On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys. 38, 447–52 (1966)

[Bell 2004] Bell, J. S.: Speakable and Unspeakable in Quantum Mechanics, 2nd edition. Cambridge University Press, Cambridge (2004)

[Brukner 2018] Brukner, Č.: A no-go theorem for observer-independent facts. Entropy 20, 350 (2018)

[Christensen 2004] Christensen, D.: Putting Logic in its Place. Oxford University Press, Oxford (2004)

[De Finetti 1937] De Finetti, B.: La prévision: ses lois logiques, ses sources subjectives. Ann. Inst. Henri Poincaré 7, 1–68 (1937)

[Earman 2018] Earman, J.: The relation between credence and chance: Lewis' "Principal Principle" is a theorem of quantum probability theory. http://philsci-archive.pitt.edu/14822/ (2018). Accessed 6 May 2019

[Feynman 1951] Feynman, R. P. The concept of probability in quantum mechanics. In Second Berkeley Symposium on Mathematical Statistics and Probability, 1950. University of California Press, Berkeley, 533–41 (1951)

[Fine 1982] Fine, A.: Joint distributions, quantum correlations, and commuting observables. J. Math. Phys. 23, 1306–10 (1982)

[Fuchs and Schack 2013] Fuchs, C. and Schack, R.: Quantum-Bayesian coherence. Rev. Mod. Phys. 85, 1693–1715 (2013)

[Gleason 1957] Gleason, A.M.: Measures on the closed subspaces of a Hilbert space. J. Math. and Mech. 6, 885–893 (1957)

[Healey 2012] Healey, R. A.: Quantum theory: A pragmatist approach. Brit. J. Phil. Sci. 63, 729–71 (2012)

[Healey 2017] Healey, R. A.: The Quantum Revolution in Philosophy. Oxford University Press, Oxford (2017)

[Healey 2018] Healey, R. A.: Quantum theory and the limits of objectivity. Found. Phys. 48, 1568–89 (2018)

[Hemmo and Pitowsky 2007] Hemmo, M. and Pitowsky, I.: Quantum probability and many worlds. Stud. Hist. Phil. Mod. Phys. 38, 333–350 (2007)

[Ismael 2008] Ismael, J. T.: Raid! Dissolving the big, bad bug. Nous 42, 292-307 (2008)

[Jaynes 2003] Jaynes, E. T.: Probability Theory: The Logic of Science, G. Larry Bretthorst (ed.) Cambridge University Press, Cambridge (2003)

[Kochen and Specker 1967] Kochen, S. and Specker, E.: The problem of hidden variables in quantum mechanics. J. Math. and Mech. 17, 59–87 (1967)

[Leegwater 2018] Leegwater, G.: When Greenberger, Horne and Zeilinger meet Wigner's friend. arXiv:1811.02442v2 [quant-ph] 7 Nov 2018. Accessed 6 May 2019

[Lewis 1980] Lewis, D. K.: A subjectivist's guide to objective chance. In R. C. Jeffrey (ed.), Studies in Inductive Logic and Probability, Volume II. University of California Press, Berkeley, 263–293 (1980)

[Lewis 1994] Lewis, D. K.: Humean supervenience debugged. Mind 103, 473-90 (1994)

[Pitowsky 2003] Pitowsky, I.: Betting on the outcomes of measurements: A Bayesian theory of quantum probability. Stud. Hist. Phil. Mod. Phys. 34, 395–414 (2003)

[Pitowski 2006] Pitowsky, I.: Quantum mechanics as a theory of probability. In W. Demopoulos, and I. Pitowsky (eds) Physical Theory and its Interpretation. Springer, Dordrecht, 213–240 (2006)

[Ramsey 1926] Ramsey, F. P.: Truth and probability. Reprinted In D. H. Mellor (ed) F. P. Ramsey: Philosophical Papers. Cambridge University Press, Cambridge 1990 (1926)

[Ruetsche 2011] Ruetsche, L.: Interpreting Quantum Theories. Oxford University Press, Oxford (2011)

[Ruetsche and Earman 2012] Ruetsche, L. and Earman, J.: Infinitely challenging: Pitowsky's subjective interpretation and the physics of infinite systems. In Y. Ben-Menahem and M. Hemmo (eds.), Probability in Physics. Springer, Berlin (2012)